Homework 1: Vector Analysis
Due Wednesday, August 30th

This assignment is due at our second class, on August 30th.

This is the first homework assignment for Physics 351. It is a review to make sure you are up to speed on a few topics covered in your multivariable calc and Physics 301 (Math Methods) classes. We will spend the first week reviewing some more complicated topics, but you should already know the material covered in this assignment when class starts. There are two important rules for this assignment:

- You may not use MATHEMATICA or any other computer algebra system on any of these problems. That includes Maple, Matlab, Wolfram Alpha, Sage, Cadabra, etc. You may not consult ChatGPT, Google Bard, or *any* LLM or "AI" assistant. Everything must be done by hand.
- You should do these problems on your own. On future assignments I'll encourage you to collaborate with classmates, but I want you to do this assignment on your own so you can assess whether or not you are familiar with the material.

Think *very carefully* about the second point. This is a review of skills that are absolutely essential for this class. You really need to make sure you are comfortable with this stuff before we get started, and the only way to do that is to work these problems out yourself. If you have questions you can email me at rmcnees@luc.edu or stop by my office (Cudahy 314).

Problem 1: Div, Grad, and Curl Identities

Let $f(x,y,z)$ and $g(x,y,z)$ be scalar functions, and let $\vec{A}(x,y,z)$ and $\vec{B}(x,y,z)$ be vector functions. Prove the following identities involving divergence, gradient, and curl.

$$\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g \tag{1}$$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B} \tag{2}$$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B} \tag{3}$$

$$\vec{\nabla}(fg) = g\,\vec{\nabla}f + f\,\vec{\nabla}g \tag{4}$$

$$\vec{\nabla}\cdot(f\,\vec{A}) = f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f \tag{5}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \tag{6}$$

$$\vec{\nabla}\times(f\,\vec{A}) = \vec{\nabla}f\times\vec{A} + f\,\vec{\nabla}\times\vec{A} \tag{7}$$

In this case, "prove" means you should show enough steps so that someone looking over your work can tell that you convinced yourself that each one of these identities is correct.

Problem 2: Divergence and Curl

Compute the divergence and curl of the following vectors:

(a) $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$.

(b) The unit vector $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$

(c) $\vec{V} = z\,\hat{x} + x\,\hat{y} + y\,\hat{z}$

(d) $\vec{w} = -x^2\,y\,\hat{x} + y^2\,x\,\hat{y} + x\,y\,z\,\hat{z}$

Problem 3: Position Vector

Suppose $\vec{k} = k_x\,\hat{x} + k_y\,\hat{y} + k_z\,\hat{z}$ is a constant vector (that is, k_x, k_y, and k_z are all constants) and $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$ is the position vector of a point with coordinates (x, y, z). Evaluate the following:

(a) $\vec{\nabla} \times (\,\vec{k} \times \vec{r}\,)$

(b) $\vec{\nabla}(\,\vec{k} \cdot \vec{r}\,)$

Problem 4: Line Integrals

Given the vector $\vec{V} = x\,y\,\hat{x} + y^2\,\hat{y}$, evaluate the line integral

$$\int_{(0,\,0)}^{(2,\,8)} \mathrm{d}\vec{\ell} \cdot \vec{V} \tag{8}$$

from the point $(0, 0)$ to the point $(2, 8)$, along the following paths:

(a) $y = x^2 + 2\,x$

(b) Along $y = 0$ from $x = 0$ to $x = 2$, then along $x = 2$ from $y = 0$ to $y = 8$.

(c) Along $x = 0$ from $y = 0$ to $y = 8$, then along $y = 8$ from $x = 0$ to $x = 2$.

In this problem we're considering paths in the x-y plane; there is no z direction to worry about. If you don't yet have a copy of the textbook, $\mathrm{d}\vec{\ell}$ is the notation Griffiths uses for the infinitesimal displacement vector $\mathrm{d}x\,\hat{x} + \mathrm{d}y\,\hat{y}$, so $\mathrm{d}\vec{\ell} \cdot \vec{V} = V_x\,dx + V_y\,dy$. **Remember**: this is a *line* integral. Take the time to review line, surface, and volume integrals if you aren't clear on what each one is and how they are different! There are a few sets of review notes on line integrals available here.

*Problem 5: Building a Plane out of a Point and two Vectors

Suppose $\vec{A}$ and $\vec{B}$ are two vectors that point in different directions. We can always find a plane that is parallel to both vectors. If a vector $\vec{n}$ is normal (orthogonal, perpendicular) to both $\vec{A}$ and $\vec{B}$, then it is normal to any plane parallel to $\vec{A}$ and $\vec{B}$. There are many such planes, but only one of them will pass through a given point **P**. Give an equation describing the plane parallel to the vectors $\vec{A} = -1\,\hat{x} - 2\,\hat{y} + 1\,\hat{z}$ and $\vec{B} = -1\,\hat{x} + 3\,\hat{y} - 4\,\hat{z}$, that passes through the point $(1, 1, 2)$. This means a condition (equation) satisfied by the coordinates (x, y, z) of any point that is in the plane.

Hint: We already know that $\vec{n}$ is perpendicular to the vectors $\vec{A}$ and $\vec{B}$. It should also be perpendicular to the separation vector between **P** and any other point (x, y, z) in the plane.